Exercise 12

Find the solution of the dissipative wave equation

$$u_{tt} - c^2 u_{xx} + \alpha u_t = 0, \quad -\infty < x < \infty,\ t > 0,$$
$$u(x,0) = f(x), \quad \left(\frac{\partial u}{\partial t}\right)_{t=0} = g(x) \quad \text{for } -\infty < x < \infty,$$

where $\alpha > 0$ is the dissipation parameter.

Solution

The PDE is defined for $-\infty < x < \infty$, so we can apply the Fourier transform to solve it. We define the Fourier transform here as

$$\mathcal{F}\{u(x,t)\} = U(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} u(x,t)\, dx,$$

which means the partial derivatives of u with respect to x and t transform as follows.

$$\mathcal{F}\left\{\frac{\partial^n u}{\partial x^n}\right\} = (ik)^n U(k,t)$$
$$\mathcal{F}\left\{\frac{\partial^n u}{\partial t^n}\right\} = \frac{d^n U}{dt^n}$$

Take the Fourier transform of both sides of the PDE.

$$\mathcal{F}\{u_{tt} - c^2 u_{xx} + \alpha u_t\} = \mathcal{F}\{0\}$$

The Fourier transform is a linear operator.

$$\mathcal{F}\{u_{tt}\} - c^2 \mathcal{F}\{u_{xx}\} + \alpha \mathcal{F}\{u_t\} = 0$$

Transform the derivatives with the relations above.

$$\frac{d^2U}{dt^2} - c^2 (ik)^2 U + \alpha \frac{dU}{dt} = 0$$

Expand the coefficient of U.

$$\frac{d^2U}{dt^2} + \alpha \frac{dU}{dt} + c^2 k^2 U = 0 \tag{1}$$

The PDE has thus been reduced to an ODE. Before we solve it, we have to transform the initial conditions as well. Taking the Fourier transform of the initial conditions gives

$$u(x,0) = f(x) \quad \rightarrow \quad \mathcal{F}\{u(x,0)\} = \mathcal{F}\{f(x)\}$$
$$U(k,0) = F(k) \tag{2}$$
$$\frac{\partial u}{\partial t}(x,0) = g(x) \quad \rightarrow \quad \mathcal{F}\left\{\frac{\partial u}{\partial t}(x,0)\right\} = \mathcal{F}\{g(x)\}$$
$$\frac{dU}{dt}(k,0) = G(k). \tag{3}$$

Equation (1) is an ODE in t, so k is treated as a constant. We can solve it with the Laplace transform since $t > 0$. The Laplace transform of $U(k,t)$ is defined as

$$\mathcal{L}\{U(k,t)\} = \overline{U}(k,s) = \int_0^\infty e^{-st} U(k,t)\, dt,$$

so the first and second derivatives transform as follows.

$$\mathcal{L}\left\{\frac{dU}{dt}\right\} = s\overline{U}(k,s) - U(k,0) \tag{4}$$

$$\mathcal{L}\left\{\frac{d^2U}{dt^2}\right\} = s^2\overline{U}(k,s) - sU(k,0) - \frac{dU}{dt}(k,0) \tag{5}$$

Take the Laplace transform of both sides of equation (1).

$$\mathcal{L}\left\{\frac{d^2U}{dt^2} + \alpha\frac{dU}{dt} + c^2k^2U\right\} = \mathcal{L}\{0\}$$

The Laplace transform is a linear operator.

$$\mathcal{L}\left\{\frac{d^2U}{dt^2}\right\} + \alpha\mathcal{L}\left\{\frac{dU}{dt}\right\} + c^2k^2\mathcal{L}\left\{U\right\} = 0$$

Use equations (4) and (5) here.

$$\left[s^2\overline{U}(k,s) - sU(k,0) - \frac{dU}{dt}(k,0)\right] + \alpha[s\overline{U}(k,s) - U(k,0)] + c^2k^2\overline{U}(k,s) = 0$$

Expand the left side and substitute equations (2) and (3).

$$s^2\overline{U}(k,s) - sF(k) - G(k) + \alpha s\overline{U}(k,s) - \alpha F(k) + c^2k^2\overline{U}(k,s) = 0$$

The ODE has thus been reduced to an algebraic equation. Factor $\overline{U}(k,s)$ and bring the terms without it to the right side.

$$(s^2 + \alpha s + c^2k^2)\overline{U}(k,s) = sF(k) + G(k) + \alpha F(k)$$

Factor $F(k)$ on the right side and divide both sides by $s^2 + \alpha s + c^2k^2$ to solve for $\overline{U}$.

$$\overline{U}(k,s) = \frac{F(k)(s+\alpha) + G(k)}{s^2 + \alpha s + c^2k^2}.$$

In order to change back to $u(x,t)$, we have to take the inverse Laplace transform of $\overline{U}(k,s)$ to get $U(k,t)$ and then take the inverse Fourier transform of it. Our task now is to write $\overline{U}$ in a form that we can easily transform. The two inverse Laplace transforms we will eventually use are

$$\mathcal{L}^{-1}\left\{\frac{s-a}{(s-a)^2 + b^2}\right\} = e^{at}\cos bt \tag{6}$$

$$\mathcal{L}^{-1}\left\{\frac{b}{(s-a)^2 + b^2}\right\} = e^{at}\sin bt, \tag{7}$$

so we want to write $\overline{U}$ with terms that have these forms. Complete the square in the denominator.

$$\overline{U}(k,s) = \frac{F(k)(s+\alpha) + G(k)}{\left(s + \frac{\alpha}{2}\right)^2 + \left(c^2k^2 - \frac{\alpha^2}{4}\right)}$$

Split up the fraction into three.

$$\bar{U}(k,s) = \frac{\left(s+\frac{\alpha}{2}\right)F(k)}{\left(s+\frac{\alpha}{2}\right)^2 + \left(c^2k^2 - \frac{\alpha^2}{4}\right)} + \frac{\alpha}{2}\frac{F(k)}{\left(s+\frac{\alpha}{2}\right)^2 + \left(c^2k^2 - \frac{\alpha^2}{4}\right)} + \frac{G(k)}{\left(s+\frac{\alpha}{2}\right)^2 + \left(c^2k^2 - \frac{\alpha^2}{4}\right)}$$

Multiply the numerator and denominator of the second and third fractions by $\sqrt{c^2k^2 - \alpha^2/4}$.

$$\begin{aligned}\bar{U}(k,s) = \frac{\left(s+\frac{\alpha}{2}\right)F(k)}{\left(s+\frac{\alpha}{2}\right)^2 + \left(c^2k^2 - \frac{\alpha^2}{4}\right)} + \frac{\alpha}{2}\frac{F(k)}{\sqrt{c^2k^2 - \frac{\alpha^2}{4}}}\frac{\sqrt{c^2k^2 - \frac{\alpha^2}{4}}}{\left(s+\frac{\alpha}{2}\right)^2 + \left(c^2k^2 - \frac{\alpha^2}{4}\right)} \\ + \frac{G(k)}{\sqrt{c^2k^2 - \frac{\alpha^2}{4}}}\frac{\sqrt{c^2k^2 - \frac{\alpha^2}{4}}}{\left(s+\frac{\alpha}{2}\right)^2 + \left(c^2k^2 - \frac{\alpha^2}{4}\right)}\end{aligned}$$

Now we're ready to take the inverse Laplace transform. Use equations (6) and (7) here.

$$\begin{aligned}U(k,t) = F(k)e^{-\frac{\alpha}{2}t}\cos\sqrt{c^2k^2 - \frac{\alpha^2}{4}}t + \frac{\alpha}{2}\frac{F(k)}{\sqrt{c^2k^2 - \frac{\alpha^2}{4}}}e^{-\frac{\alpha}{2}t}\sin\sqrt{c^2k^2 - \frac{\alpha^2}{4}}t \\ + \frac{G(k)}{\sqrt{c^2k^2 - \frac{\alpha^2}{4}}}e^{-\frac{\alpha}{2}t}\sin\sqrt{c^2k^2 - \frac{\alpha^2}{4}}t\end{aligned}$$

To make $U(k,t)$ easier to work with, introduce a new variable $\omega = \omega(k)$ for the square root term.

$$\omega(k) = \sqrt{c^2k^2 - \frac{\alpha^2}{4}}$$

Then, after factoring,

$$U(k,t) = \left[F(k)\left(\cos\omega t + \frac{\alpha}{2\omega}\sin\omega t\right) + \frac{G(k)}{\omega}\sin\omega t\right]e^{-\frac{\alpha}{2}t}.$$

It's not necessary to consider the case where $c^2k^2 - \frac{\alpha^2}{4} < 0$ because $\cos i\omega t = \cosh\omega t$ and $-i\sin i\omega t = \sinh\omega t$. We're ready now to take the inverse Fourier transform. It is defined as

$$\mathcal{F}^{-1}\{U(k,t)\} = u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} U(k,t)e^{ikx}\,dk.$$

Plug $U(k,t)$ into the definition of the inverse Fourier transform to get $u(x,t)$.

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[F(k)\left(\cos\omega t + \frac{\alpha}{2\omega}\sin\omega t\right) + \frac{G(k)}{\omega}\sin\omega t\right]e^{-\frac{\alpha}{2}t}e^{ikx}\,dk$$

Recall that sine and cosine can be written in terms of exponentials using Euler's formula.

$$\begin{aligned}\cos\omega t &= \frac{e^{i\omega t} + e^{-i\omega t}}{2} \\ \sin\omega t &= \frac{e^{i\omega t} - e^{-i\omega t}}{2i}\end{aligned}$$

Substituting these expressions and bringing the $e^{-\frac{\alpha}{2}t}$ in front of the integral, we get

$$u(x,t) = \frac{e^{-\frac{\alpha}{2}t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[F(k) \left(\frac{e^{i\omega t} + e^{-i\omega t}}{2} + \frac{\alpha}{2\omega} \frac{e^{i\omega t} - e^{-i\omega t}}{2i} \right) + \frac{G(k)}{\omega} \frac{e^{i\omega t} - e^{-i\omega t}}{2i} \right] e^{ikx}\, dk.$$

Expand the integrand and factor the terms in $e^{i\omega t}$ and $e^{-i\omega t}$.

$$u(x,t) = \frac{e^{-\frac{\alpha}{2}t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left\{ \left[\frac{F(k)}{2} + \frac{\alpha}{2\omega} F(k) \frac{1}{2i} + \frac{G(k)}{2i\omega} \right] e^{i\omega t} + \left[\frac{F(k)}{2} - \frac{\alpha}{2\omega} F(k) \frac{1}{2i} - \frac{G(k)}{2i\omega} \right] e^{-i\omega t} \right\} e^{ikx}\, dk$$

Factor the terms in square brackets and distribute e^{ikx}.

$$u(x,t) = \frac{e^{-\frac{\alpha}{2}t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left\{ \frac{1}{2} \left[F(k) \left(1 + \frac{\alpha}{2i\omega} \right) + \frac{G(k)}{i\omega} \right] e^{i(kx+\omega t)} + \frac{1}{2} \left[F(k) \left(1 - \frac{\alpha}{2i\omega} \right) - \frac{G(k)}{i\omega} \right] e^{i(kx-\omega t)} \right\} dk$$

Therefore,

$$u(x,t) = \frac{e^{-\frac{\alpha}{2}t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[A(k) e^{i(kx+\omega t)} + B(k) e^{i(kx-\omega t)} \right] dk,$$

where

$$\begin{aligned}
\omega = \omega(k) &= \sqrt{c^2k^2 - \frac{\alpha^2}{4}} \\
A(k) &= \frac{1}{2} \left[F(k) \left(1 + \frac{\alpha}{2i\omega} \right) + \frac{G(k)}{i\omega} \right] \\
B(k) &= \frac{1}{2} \left[F(k) \left(1 - \frac{\alpha}{2i\omega} \right) - \frac{G(k)}{i\omega} \right] \\
F(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x)\, dx \\
G(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} g(x)\, dx.
\end{aligned}$$

Perhaps the most striking feature of the solution is the factor $e^{-\frac{\alpha}{2}t}$, which means that the amplitude of the wave decreases exponentially with time. This is due to the dissipation parameter α in the PDE. In the event $\alpha = 0$, then

$$\begin{aligned}
\omega &= ck \\
A(k) &= \frac{1}{2} \left[F(k) + \frac{G(k)}{ick} \right] \\
B(k) &= \frac{1}{2} \left[F(k) - \frac{G(k)}{ick} \right],
\end{aligned}$$

and d'Alembert's solution for the wave equation is obtained as expected (see pg. 37 in the textbook).